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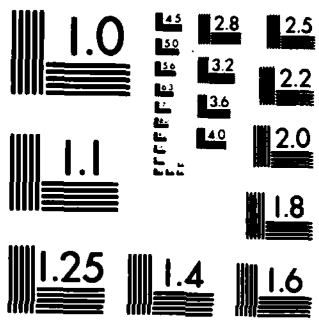

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A GENERAL MINIMAL REPAIR
MAINTENANCE MODEL

by

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A GENERAL MINIMAL REPAIR MAINTENANCE MODEL

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ABSTRACT

A general model is introduced which incorporates minimal repair, planned and unplanned replacements, and costs which depend on time. Finite and infinite horizon results are obtained. Various special cases are considered. Furthermore a shock model with general cost structure is considered.

AMS 1970 Subject Classification: Primary 62N05; Secondary 60K10.

Key Words: Maintenance policies, minimal repair, renewal reward, finite and infinite horizon, shock models.

1. Introduction

Most of the literature concerning the stochastic behavior of repairable systems assumes that when a unit is repaired, it becomes as good as new. Although this assumption provides a technical convenience, in many situations it is not realistic. A more realistic situation is that when the unit is repaired, it is restored to its functioning condition just prior to failure. This is called minimal repair. Several authors have discussed this situation. See for example Ascher and Feingold (1979) and the references contained in that paper.

Barlow and Hunter (1960) considered minimal repair in the case of periodic replacement or overhaul at times T,2T,3T,... (for some $T > 0$) and minimal repair if the system failed otherwise. They considered costs c_2 of replacement and c_1 for each minimal repair. This model has been generalized by Beichelt (1976), Boland and Proschan (1982) and Boland (1982).

In this paper a general model which incorporates minimal repair, planned and unplanned replacements, and costs which depend on time. The model is described explicitly at the beginning of the next section. The expected long run average cost is derived for this model and optimization results are obtained in both the finite and infinite horizon case. A shock model similar to the one of Boland and Proschan (1983) is proposed, but with general cost structure. As special cases, various results from Barlow and Proschan (1965) are obtained as well as many of the results of Beichelt (1976), Boland and Proschan (1982) and Boland (1982).

In Section 2, the model is described, then the total expected long run cost per unit time is found. Theorem 2.10 gives a general optimization result for the infinite horizon case. Theorem 2.13, 2.14 and Corollary 2.15 give finite horizon result. Also various special cases are detailed.

Section 3 contains the shock model with general cost structure.

2. General model

We consider a preventative maintenance model in which minimal repair or replacement takes place according to the following scheme. An operating unit is completely replaced whenever it reaches age T $(T > 0)$ at a cost c_0 (planned replacement). If it fails at age $y < T$, it is either replaced by a new unit with probability $p(y)$ at a cost c_∞ (unplanned replacement), or it undergoes *minimal* repair with probability $q(y) = 1 - p(y)$. The cost of the i-th minimal repair is $c_i(y)$. After a complete replacement (i.e., an unplanned or planned replacement by a new unit), the procedure is repeated. We assume all failures are instantly detected and repaired. We always assume $c_0 > 0$.

The survival distribution $\bar{F}(y)$ of a unit is assumed to be a continuous function of y. If no planned replacements are considered (i.e., $T = \infty$), the survival distribution of the time between successive unplanned replacements is given by

$$(2.1) \qquad \bar{F}_p(y) = \exp\{-\int_0^y p(x)\bar{F}^{-1}(x)dF(x)\}.$$

See Block, Borges, and Savits (BBS) (1982) for a derivation of this result.

Let $Y_1, Y_2, \ldots$ be iid random variables with survival distribution $\bar{F}_p$ and set $Y_i^* = Y_i \wedge T$ (where $a \wedge b = \min(a,b)$) for $i = 1,2,\ldots$. Let R_i^* denote the operational cost over the renewal interval Y_i^*. Thus $\{(Y_i^*, R_i^*)\}$ constitutes a renewal reward process. If K(t) denotes the expected cost of operating the system over the time interval $[0,t]$, then it is well known that

$$(2.2) \qquad \lim_{t\to\infty} \frac{K(t)}{t} = \frac{E[R_1^*]}{E[Y_1^*]}.$$

(See, e.g., Ross (1970), page 52). We shall denote the right-hand side of

(2.2) by B(T).

Since $E[Y_1^*] = \int_0^T \bar{F}_p(t)dt$, we need only evaluate $E[R_1^*]$. Using the techniques of BSS (1982), one obtains that

$$(2.3)\quad E[R_1^*] = c_\infty F_p(T) + c_0 \bar{F}_p(T) + \int_0^T q(y)h(y)\exp\{\int_0^y q(z)\bar{F}^{-1}(z)dF(z)\}dF(y),$$

where

$$(2.4)\quad h(y) = \sum_{j=0}^{\infty} c_{j+1}(y)\ \exp\{-\int_0^y q(u)\bar{F}^{-1}(u)dF(u)\}\ \frac{1}{j!}\ (\int_0^y q(v)\bar{F}^{-1}(v)dF(v))^j.$$

The tedious but routine details are contained in the Appendix. If M(y) denotes a nonhomogeneous Poisson process with mean $\int_0^y q(u)\bar{F}^{-1}(u)dF(u)$, then we can also write (2.4) as

$$(2.5)\qquad h(y) = E[c_{M(y)+1}(y)].$$

(2.6) <u>Remark</u>. We shall only consider cost structures such that h(y) is finite for all $y \geq 0$.

2.1 Infinite horizon.

For the infinite horizon case we want to find a T which minimizes B(T), the total expected long run cost per unit time. Recall that

$$(2.7)\quad B(T) = \frac{c_\infty F_p(T) + c_0\bar{F}_p(T) + \int_0^T q(y)h(y)\ \exp\{\int_0^y q(z)\bar{F}^{-1}(z)dF(z)\}dF(y)}{\int_0^T \bar{F}_p(y)dy}.$$

We now assume that F is absolutely continuous with failure rate function r and that the functions r, p and h are continuous. In this case we can differentiate B with respect to T. Hence $\frac{dB}{dT} = 0$ if and only if

$$(2.8)\qquad 0 = \{\int_0^T [p(T)r(T) - p(y)r(y)]\bar{F}_p(y)dy][c_\infty - c_0 - h(T)] - c_0$$

$$+ \int_0^T [h(T)-h(y)]r(y)q(y)\bar{F}_p(y)dy + h(T)\int_0^T [r(T) - r(y)]\bar{F}_p(y)dy.$$

We can also rewrite the above as

$$(2.9)\qquad 0 = \{\int_0^T [p(T)r(T) - p(y)r(y)]\bar{F}_p(y)dy\}[c_\infty - c_0] - c_0$$

$$+ \int_0^T [h(T)r(T)q(T) - h(y)r(y)q(y)]\bar{F}_p(y)dy.$$

(2.10) Theorem. Let F have failure rate r and suppose that the functions r,p and h are continuous. Then if either

(a) $r, p\cdot r$ and h are increasing with r unbounded and $c_0 + h(T) < c_\infty$ for all $T \geq 0$,

or

(b) $(c_\infty - c_0)p(y)r(y) + h(y)r(y)q(y)$ increases to $+\infty$,

then there exists at least one finite positive T_0 which minimizes the total expected long run cost per unti time B(T). Furthermore if any of the functions in (a) or (b) are strictly increasing, then T_0 is unique.

Proof. If the conditions of the theorem are satisfied, then the right-hand side of (2.8)((2.9)) is a continuous increasing function of T which is negative $(-c_0)$ at $T = 0$ and tends to $+\infty$ as $T \to +\infty$. Hence there is at least one value $\infty > T_0 > 0$ which satisfies (2.8)((2.9)). Since B'(T) has the same sign change pattern (-,0,+), it follows that B has a minimum at T_0. Under the strict increasing assumption, B(T) is strictly increasing and so T_0 is unique.

(2.11) Remark. If $c_j(y)$ increases in j and y, then h(y) is increasing. If

also c_j is continuous and $h(y)$ is finite for all $y \geq 0$, then $h(y)$ is continuous.

(2.12) Special cases:

Case 1 ($p \equiv 0$). This case corresponds to the situation of minimal repairs with no unplanned replacement. It is more general than the model considered by Boland (1982) since the cost of minimal repairs $c_j(y)$ depends on the number j as well as the age y. Now (2.7) reduces to

$$B(T) = T^{-1}[c_0 + \int_0^T h(y)\bar{F}^{-1}(y)dF(y)]$$

and (2.9) becomes

$$c_0 = \int_0^T [h(T)r(T) - h(y)r(y)]dy.$$

Thus a minimum exists if $h(y)r(y)$ increases to $+\infty$.

Case 1a ($p \equiv 0$, $c_j(t) = c(t)$). *This was the case considered by* Boland (1982). Here $h(y) = c(y)$. One special case where the condition in the theorem holds is when $c(t)$ is increasing to ∞ and F is IFR.

Case 1b ($p \equiv 0$, $c_j(t) = c_j$). Boland and Proschan (1982) investigated this case. In particular they considered the cost structure $c_j = a + jc$.

Case 2 ($p \equiv 1$). This corresponds to no minimal repair (i.e., only complete replacements) and reduces to the classical age replacement situation considered by Barlow and Proschan (1965). Now (2.7) becomes

$$B(T) = \frac{c_\infty F(T) + c_0 \bar{F}(T)}{\int_0^T \bar{F}(y)dy}$$

and (2.9) reduces to

$$c_0 = (c_\infty - c_0)\int_0^T [r(T)-r(y)]\bar{F}(y)dy.$$

Hence if $(c_\infty - c_0)r(y)$ increases to $+\infty$, we obtain a minimum. Barlow and Proschan (1965) have the equivalent condition that $c_0 < c_\infty$ and $r(y)$ increases to ∞.

<u>Case 3</u> ($p(t) \equiv p$, $0<p<1$). In this case we are assuming that the probability of a complete unplanned repair (and consequently minimal repair) does not depend on age. Thus (2.7) is

$$B(T) = \frac{c_\infty[1-\bar{F}^p(T)] + c_0\bar{F}^p(T) + q\int_0^T h(y)\bar{F}^q(y)dF(y)}{\int_0^T \bar{F}^p(y)dy}$$

and (2.9) becomes

$$c_0 = p(c_\infty - c_0)\int_0^T [r(T)-r(y)]\bar{F}^p(y)dy + q\int_0^T [h(T)r(T) - h(y)r(y)]\bar{F}^p(y)dy.$$

Therefore, if $\{p(c_\infty - c_0)r(y) + q\, h(y)r(y)\}$ increases to $+\infty$, we again achieve a minimum.

<u>Case 4</u> ($c_j(t) \equiv c$). Here the minimal repair costs depend on neither age nor number and so $h(y) \equiv c$. Beichelt (1976) considered this case. Equation (2.7) becomes

$$B(T) = \frac{c_\infty F_p(T) + c_0\bar{F}_p(T) + c\int_0^T q(y)\ \exp\{\int_0^y q(z)\bar{F}^{-1}(z)dF(z)\}dF(y)}{\int_0^T \bar{F}_p(y)dy}$$

$$= \frac{(c_\infty - c_0)F_p(T) + c_0\bar{F}_p(T) + c\int_0^T \bar{F}_p(y)\bar{F}^{-1}(u)dF(y)}{\int_0^T \bar{F}_p(y)dy}$$

which is equation (6) of Beichelt. Equation (2.8) takes the form

$$c_0 = (c_\infty - c_0 - c)\int_0^T [p(T)r(T) - p(y)r(y)]\bar{F}_p(y)dy + c\int_0^T [r(T) - r(y)]\bar{F}_p(y)dy.$$

Hence if $\{(c_\infty - c_0 - c)p(y)r(y) + c\, r(y)\}$ increases to $+\infty$, then a minimum exists. A special case occurs when $c_0 + c < c_\infty$ and r and p are increasing with r unbounded.

2.2 Finite horizon.

In this section we consider the problem of finding the minimum cost over a fixed time horizon $[0,s)$. Since the general problem is quite involved, we only consider the case $p \equiv 0$; i.e., no unplanned replacements. This extends the results of Boland and Proschan (1982).

For the case $p \equiv 0$, one can show using the techniques of BSS (1982)(as in the Appendix) that the expected cost on $[0,t]$ due to minimal repairs only is given by

$$\int_0^t h(y)\bar{F}^{-1}(y)\,dF(y),$$

where $h(y)$ is as given in (2.4) or (2.5) with $q \equiv 1$; i.e.,

$$h(y) = \sum_{j=0}^{\infty} c_{j+1}(y)\; e^{-R(y)}\; \frac{R^j(y)}{j!}$$

and $R(y) = -\log \bar{F}(y)$ is the hazard function of F. Again we only consider cost structures such that $h(y) < \infty$ for all $0 \le y \le s$. We will assume without loss of generality that the original cost of the system is zero.

(2.13) Theorem. The expected cost in $[0,s)$ of our model with $p \equiv 0$ is given by

$$K_s(T) = \begin{cases} (k-1)c_0 + k\displaystyle\int_0^T h(y)\bar{F}^{-1}(y)\,dF(y) & \text{if } kT = s \\ k\,c_0 + k\displaystyle\int_0^T h(y)\bar{F}^{-1}(y)\,dF(y) + \int_0^{s-kT} h(y)\bar{F}^{-1}(y)\,dF(y) & \text{if } kT < s < (k+1)T. \end{cases}$$

Furthermore K_s is continuous on $[0,s]$ except possibly at the points $s, s/2, s/3, \ldots$ and right-continuous at these points. Note that for $T = s$ the cost on $[0,s)$ does not include the planned replacement cost.

Proof. From our original assumption that F is continuous, the function $\int_0^T h(y)\bar{F}^{-1}(y)\,dF(y)$ is continuous. The result follows easily from this.

(2.14) <u>Theorem</u>. Let F have failure rate r and suppose that the function $r \cdot h$ is increasing and continuous on [0,s]. Then $K_s(T)$ is minimized on [0,s] at one of the points $s, s/2, s/3, \ldots$.

<u>Proof</u>. For $T \in (s/k+1, s/k)$, we have

$$\frac{dK_s(T)}{dT} = k[h(T)r(T) - h(s-kT)r(s-kT)].$$

Thus, by our assumptions, $\frac{dK_s(T)}{dT}$ is nonnegative and the conclusion follows.

(2.15) <u>Corollary</u>. Let F be IFR with a continuous failure rate r on [0,s]. If the functions $c_j(u)$ are continuous and increasing in j and u for $0 \le u \le s$, then the results of Theorem 2.14 hold.

<u>Proof</u>. One can show that our assumptions on $c_j(u)$ imply that h(y) is increasing and continuous in $0 \le y \le s$. See Remark (2.11).

3. Shock model

In Boland and Proschan (1983) a system which is subject to shocks is considered. Under a simple cost structure the problem of optimum replacements is considered. In this section we examine a similar system but with general costs which depend on the number of shock and the time at which it occurs.

We consider a system which is subjected to shocks at times $S_1, S_2, \ldots$ according to a nonhomogeneous Poisson process $\{N(t), t \ge 0\}$ having mean function $\Lambda(t)$. The cost of operating the system per unit time in $[S_{j-1}, S_j)$ is $c_j(u)$ for $u \ge 0$ and $j = 1,2,\ldots$. The system is periodically replaced at times $T, 2T, \ldots$ at a fixed cost of c_0 and the shock process resets to zero at each of these replacements.

If k shocks occur in $[0,T)$, the cost of operating the system is given by

$$C(T) = \sum_{j=0}^{k-1} \int_{S_j}^{S_{j+1}} c_{j+1}(u)\,du + \int_{S_k}^{T} c_{k+1}(u)\,du = \sum_{j=0}^{\infty} \int_{S_j \wedge T}^{S_{j+1}\wedge T} c_{j+1}(u)\,du.$$

Thus the expected cost is

$$K(T) = E[C(T)] = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} E\Big[\int_{S_j\wedge T}^{S_{j+1}\wedge T} c_{j+1}(u)\,du;\ N(T)=k\Big]$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} E\Big[\int_0^{\infty} c_{j+1}(u) I_{[S_j\wedge T, S_{j+1}\wedge T)}(u)\,du;\ N(T)=k\Big]$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \int_0^{\infty} c_{j+1}(u) E[I_{[S_j\wedge T, S_{j+1}\wedge T)}(u);\ N(T)=k]\,du.$$

For $j = 0,1,\ldots,k-1$ and $0 \le u < T$,

$$E[I_{[S_j\wedge T, S_{j+1}\wedge T)}(u); N(T)=k] = P[S_j \le u < S_{j+1}; N(T)=k]$$

$$= P[N(u)=j; N(T)-N(u)=k-j],$$

and for j 'k, $0 \le u < T$,

$$E[I_{[S_k\wedge T, S_{k+1}\wedge T)}(u); N(T)=k] = P[S_k \le u < T < S_{k+1}]$$

$$= P[N(u)=k, N(T)-N(u)=0].$$

Thus

$$K(T) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \int_0^T c_{j+1}(u)\ P[N(u)=j]P[N(T)-N(u)=k-j]\,du$$

$$= \int_0^T \sum_{j=0}^{\infty} c_{j+1}(u)\ P[N(u)=j] \sum_{k=j}^{\infty} P[N(T)-N(u)=k-j]\,du$$

$$= H(T)$$

where $h(u) = \sum_{j=0}^{\infty} c_{j+1}(u)\ P[N(u)=j] = E[c_{N(u)+1}(u)]$ and $H(T) = \int_0^T h(u)du$.

3.1 Infinite horizon.

We consider the process on $[0,s)$. Again we may assume without loss of generality that the cost of the original system is zero. Then the expected cost on $[0,s]$ is given by

$$K_s(T) = \begin{cases} m[H(T)+c_0] + H(s-mT) & \text{if} \quad mT<s<(m+1)T \\ m\ H(T)+(m-1)c_0 & \text{if} \quad mT=s. \end{cases} \tag{3.1}$$

Thus the long run expected cost per unit time is

$$B(T) = \lim_{s\to\infty} \frac{K_s(T)}{s} = \frac{H(T)+c_0}{T}, \tag{3.2}$$

which is the average cost over $(0,T]$. So to minimize the long run average cost we want to minimize $B(T)$. Assuming that $h(u)$ is continuous, the condition $\frac{dB}{dT} = 0$ is equivalent to the well known condition

$$\int_0^T [h(T) - h(u)]du = c_0. \tag{3.3}$$

The following result is now easy to derive.

(3.4) Theorem. If there exists an inverval $[0,b)$, $0\le b\le\infty$, on which $h(u)$ is continuous, increasing and unbounded, then there exists a solution $0<T_0\le\infty$ to equation (3.3). Furthermore, if $h(u)$ is strictly increasing, then the solution is unique.

(3.5) Remark. As before if $c_j(u)$ increases in j and u, then $h(u)$ is increasing. If also each $c_j(u)$ is continuous, $\Lambda(u)$ is continuous, and $h(u)$ is finite for $u\ge 0$, then $h(u)$ is continuous.

3.2 Finite horizon.

As noted before, the expected cost on [0,s) is given by

$$K_s(T) = \begin{cases} m[H(T)+c_0]+H(s-mT) & \text{if } mT<s<(m+1)T \\ mH(T)+(m-1)c_0 & \text{if } mT = s. \end{cases} \tag{3.6}$$

We now want to minimize $K_s(T)$ for fixed s. The proofs of the following results are similar to those in section 2.2.

(3.7) Theorem. The expected cost $K_s(T)$ on [0,s) is given by (3.6). If H(u) is continuous on [0,s], then K_s is continuous on [0,s] except possibly at the points s,s/2,s/3,... and right-continuous at these points.

(3.8) Theorem. Assume that h(u) is continuous and increasing on [0,s]. Then $K_s(T)$ is minimized on [0,s] at one of the points s,s/2,s/3,....

(3.9) Note. The Remark (3.5) is also pertinent here.

Appendix

In this section we present the detailed calculations of the expression given by (2.3). We closely follow the techniques and notation of BBS (1982).

As in BBS (1982) we let $\{(S_n^{(1)}, Z_n^{(1)});\ n \geq 1\}$ be a sequence of random variables, where $S_n^{(1)} \geq 0$ denotes the time of the n^{th} repair and $Z_n^{(1)} = 0$ or 1 indicates a minimal or a complete repair at time $S_{n-1}^{(1)}$ ($S_0^{(1)} = 0$, $Z_1^{(1)} = 0$), respectively. We consider the process only up to the first complete repair at time $Y_1 = S_{\nu-1}^{(1)}$ where $\nu = \inf\{n:\ Z_n^{(1)} = 1\}$. Then from Theorem A.4 of BBS (1982) the $\{S_n^{(1)}\}_{n=1}^{\infty}$ are the jump times of a nonhomogeneous Poisson process with mean function $\int_0^t \bar{F}^{-1}(y)dF(y)$. Furthermore if p is such that $\int_0^{\infty} p(y)\bar{F}^{-1}(y)dF(y) = \infty$, then by Theorem A.5 of BBS (1982) Y_1 has distribution $\bar{F}_p(y)$ given by (2.1).

Since

$$E[R_1^*] = E[R_1^*;\ Y_1 < T] + E[R_1^*;\ Y_1 \geq T]$$

it suffices to compute each term separately. First, since $Y_1 = S_n^{(1)}$ if and only if $Z_1^{(1)} = \ldots = Z_n^{(1)} = 0$ and $Z_{n+1}^{(1)} = 1$, we have

$$E[R_1^*; Y_1 < T] = c_\infty F_p(T) + \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} E[c_j(S_j^{(1)});\ Y_1 = S_n^{(1)} < T]$$

$$= c_\infty F_p(T) + \sum_{n=2}^{\infty} \sum_{j=1}^{n-1} E[c_j(S_j^{(1)});\ Z_1^{(1)} = \ldots = Z_n^{(1)} = 0,\ Z_{n+1}^{(1)} = 1,\ S_n^{(1)} < T]$$

$$= c_\infty F_p(T) + \sum_{j=1}^{\infty} \sum_{n=j+1}^{\infty} E[c_j(S_j^{(1)})p(S_n^{(1)}) \prod_{i=2}^{n} q(S_{i-1}^{(1)});\ S_n^{(1)} < T].$$

where the last equality follows from (A.3) of BBS 1982). We now consider one of the summands for $n \geq j+1$. Again using (A.3) of BBS (1982)

$$E[c_j(S_j^{(1)})p(S_n^{(1)}) \prod_{i=2}^{n} q(S_{i-1}^{(1)});\ S_n < T]$$

$$= \int_0^T \bar{F}^{-1}(s_1)q(s_1)\int_{s_1}^T \bar{F}^{-1}(s_2)q(s_2)\ldots \int_{s_{j-1}}^T \bar{F}^{-1}(s_j)q(s_j)c_j(s_j)\int_{s_j}^T \bar{F}^{-1}(s_{j+1})q(s_{j+1})\ldots$$

$$\cdot \int_{s_{n-2}}^T \bar{F}^{-1}(s_{n-1})q(s_{n-1})\int_{s_{n-1}}^T p(s_n)dF(s_n)\ldots dF(s_1)$$

$$= \int_0^T p(s_n)\int_0^{s_n} \bar{F}^{-1}(s_{n-1})q(s_{n-1})\cdots\int_0^{s_{j+1}} \bar{F}^{-1}(s_j)q(s_j)c_j(s_j)\{\int_0^{s_j} \bar{F}^{-1}(s_{j-1})q(s_{j-1})\cdots$$

$$\cdot \int_0^{s_2} \bar{F}^{-1}(s_1)q(s_1)dF(s_1)\ldots dF(s_{j-1})\}dF(s_j)\ldots dF(s_n)$$

$$= \int_0^T p(s_n)\int_0^{s_n} \bar{F}^{-1}(s_{n-1})q(s_{n-1})\ldots\int_0^{s_{j+1}} \bar{F}^{-1}(s_j)q(s_j)c_j(s_j)$$

$$\cdot \{\frac{1}{(j-1)!}[\int_0^{s_j} \bar{F}^{-1}(u)q(u)dF(u)]^{j-1}\}dF(s_j)\ldots dF(s_n)$$

$$= \int_0^T p(s_n)\int_0^{s_n} \bar{F}^{-1}(s_j)q(s_j)c_j(s_j)\{\frac{1}{(j-1)!}[\int_0^{s_j} \bar{F}^{-1}(u)q(u)dF(u)]^{j-1}\}$$

$$\cdot [\int_{s_j}^{s_n} \bar{F}^{-1}(s_{j+1})q(s_{j+1})\cdots\int_{s_{n-2}}^{s_n} \bar{F}^{-1}(s_{n-1})q(s_{n-1})dF(s_{n-1})\cdots dF(s_{j+1})]dF(s_j)dF(s_n)$$

$$= \int_0^T p(s_n)\int_0^{s_n} \bar{F}^{-1}(s_j)q(s_j)c_j(s_j)\{\frac{1}{(j-1)!}[\int_0^{s_j}\bar{F}^{-1}(u)\,q(u)dF(u)]^{j-1}\}$$

$$\cdot \{\frac{1}{(n-j-1)!}[\int_{s_j}^{s_n} \bar{F}^{-1}(v)q(v)dF(v)]^{n-j-1}\}dF(s_j)dF(s_n).$$

Thus

$$\sum_{n=j+1}^{\infty} E[c_j(S_j^{(1)})p(S_n^{(1)})\prod_{i=2}^{n} q(S_{i-1}^{(1)});\; S_n^{(1)} < T]$$

$$= \int_0^T p(w)\int_0^w \bar{F}^{-1}(z)q(z)c_j(z)\{\frac{1}{(j-1)!}[\int_0^z \bar{F}^{-1}(u)q(u)dF(u)]^{j-1}\}$$

$$\cdot \exp\{\int_z^w \bar{F}^{-1}(v)q(v)dF(v)\}dF(z)dF(w)$$

and so

$$E[R_1^*;Y_1 < T] = c_\infty F_p(T) + \int_0^T p(w)\exp\{\int_0^w \bar{F}^{-1}(v)q(v)dF(v)\}$$

$$\cdot \int_0^w(\sum_{j=1}^{\infty} c_j(z)\{\frac{1}{(j-1)!}[\int_0^z \bar{F}^{-1}(u)q(u)dF(u)]^{j-1}\})\bar{F}^{-1}(z)q(z)dF(z)dF(w)$$

$$= c_\infty F_p(T) + \int_0^T p(w)\exp\{\int_0^w \bar{F}^{-1}(v)q(v)dF(v)\}\int_0^w \bar{F}^{-1}(z)q(z)h(z)dF(z)dF(w),$$

where

$$h(z) = \sum_{j=0}^{\infty} c_{j+1}(z)\exp\{-\int_0^z \bar{F}^{-1}(u)q(u)dF(u)\}\ \frac{1}{j!}\ [\int_0^z \bar{F}^{-1}(v)q(v)dF(v)]^j.$$

Similar calculations for the term $E[R_1^*;\ Y_1 \geq T]$ are given below.

$$E[R_1^*;Y_1 \geq T] = c_0\bar{F}_p(T) + \sum_{n=2}^{\infty}\sum_{j=1}^{n-1} E[c_j(S_j^{(1)});S_{n-1}^{(1)} < T \leq S_n^{(1)};\ Y_1 \geq T]$$

$$= c_0\bar{F}_p(T) + \sum_{n=2}^{\infty}\sum_{j=1}^{n-1} E[c_j(S_j^{(1)});Z_1^{(1)} = \ldots = Z_n^{(1)} = 0, S_{n-1}^{(1)} < T \leq S_n^{(1)}]$$

$$= c_0\bar{F}_p(T) + \sum_{j=1}^{\infty}\sum_{n=j+1}^{\infty} E[c_j(S_j^{(1)}) \prod_{i=2}^{n} q(S_{i-1}^{(1)});\ S_{n-1}^{(1)} < T \leq S_n^{(1)}].$$

For $n \geq j+1$,

$$E[c_j(S_j^{(1)}) \prod_{i=2}^{n} q(S_{i-1}^{(1)});\ S_{n-1}^{(1)} < T \leq S_n^{(1)}]$$

$$= \int_0^T \bar{F}^{-1}(s_1)q(s_1)\int_{s_1}^T \bar{F}^{-1}(s_2)q(s_2)\ldots\int_{s_{j-1}}^T \bar{F}^{-1}(s_j)q(s_j)c_j(s_j)\int_{s_j}^T \bar{F}^{-1}(s_{j+1})q(s_{j+1})\cdots$$

$$\cdot \int_{s_{n-2}}^T \bar{F}^{-1}(s_{n-1})q(s_{n-1})\int_T^{\infty} dF(s_n)dF(s_{n-1})\ldots dF(s_1)$$

$$= \bar{F}(T)\int_0^T \bar{F}^{-1}(s_{n-1})q(s_{n-1})\int_0^{s_{n-1}} \bar{F}^{-1}(s_{n-2})\, q(s_{n-2})\ldots\int_0^{s_{j+1}} \bar{F}^{-1}(s_j)q(s_j)c_j(s_j)$$

$$\cdot [\int_0^{s_j} \bar{F}^{-1}(s_{j-1})q(s_{j-1})\ldots\int_0^{s_2} \bar{F}^{-1}(s_1)q(s_1)]dF(s_1)\ldots dF(s_{n-1})$$

$$= \bar{F}(T)\int_0^T \bar{F}^{-1}(s_{n-1})q(s_{n-1})\ldots\int_0^{s_{j+1}} \bar{F}^{-1}(s_j)q(s_j)c_j(s_j)$$

$$\cdot \{\frac{1}{(j-1)!}\ [\int_0^{s_j} \bar{F}^{-1}(u)q(u)dF(u)]^{j-1}\}dF(s_j)\ldots dF(s_{n-1})$$

$$= \bar{F}(T)\int_0^T \bar{F}^{-1}(s_j)q(s_j)c_j(s_j)\{\frac{1}{(j-1)!}\ [\int_0^{s_j} \bar{F}^{-1}(u)q(u)dF(u)]^{j-1}\}$$

$$\cdot [\int_{s_j}^T \bar{F}^{-1}(s_{j+1})q(s_{j+1})\ldots\int_{s_{n-2}}^T \bar{F}^{-1}(s_{n-1})q(s_{n-1})dF(s_{n-1})\ldots dF(s_{j+1})]dF(s_j)$$

$$= \bar{F}(T) \int_0^T \bar{F}^{-1}(s_j)q(s_j)c_j(s_j)\{\frac{1}{(j-1)!}\,[\int_0^{s_j} \bar{F}^{-1}(u)q(u)dF(u)]^{j-1}\}$$

$$\cdot \{\frac{1}{(n-j-1)!}\,[\int_{s_j}^T \bar{F}^{-1}(v)q(v)dF(v)]^{n-j-1}\}dF(s_j).$$

Consequently,

$$E[R_1^*;Y_1 \geq T] = c_0\bar{F}_p(T)+\bar{F}(T)\exp\{\int_0^T \bar{F}^{-1}(v)q(v)dF(v)\}\int_0^T \bar{F}^{-1}(z)q(z)h(z)dF(z),$$

and so

$$E[R_1^*] = c_\infty F_p(T) + c_0\bar{F}_p(T)$$

$$+ \int_0^T p(w)\ \exp\{\int_0^w \bar{F}^{-1}(v)q(v)dF(v)\}\int_0^w \bar{F}^{-1}(z)q(z)h(z)dF(z)dF(w)$$

$$+ \bar{F}(T)\ \exp\{\int_0^T \bar{F}^{-1}(v)q(v)dF(v)\}\int_0^T \bar{F}^{-1}(z)q(z)h(z)dF(z).$$

But since $\exp\{\int_0^w \bar{F}^{-1}(v)q(v)dF(v)\} = \bar{F}^{-1}(w)\ \exp\{-\int_0^w \bar{F}^{-1}(y)p(y)\,dF(y)$, we obtain

$$\int_0^T p(w)\exp\{\int_0^w \bar{F}^{-1}(v)\,q(v)dF(v)\}\int_0^w \bar{F}^{-1}(z)\,q(z)h(z)dF(z)$$

$$= \int_0^T \int_z^T \bar{F}^{-1}(z)\,q(z)h(z)p(w)\ \exp\{\int_0^w \bar{F}^{-1}(y)\,q(v)dF(v)\}dF(w)dF(z)$$

$$= \int_0^T \bar{F}^{-1}(z)\,q(z)h(z)[\int_z^T \bar{F}^{-1}(w)p(w)\exp\{-\int_0^w \bar{F}^{-1}(y)p(y)dF(y)\}dF(w)]dF(z)$$

$$= \int_0^T \bar{F}^{-1}(z)q(z)h(z)\ \ [\exp\{-\int_0^z \bar{F}^{-1}(y)p(y)dF(y)\}-\exp\{-\int_0^T \bar{F}^{-1}(y)p(y)dF(y)\}]dF(z)$$

$$= \int_0^T \bar{F}^{-1}(z)q(z)h(z)\exp\{-\int_0^z \bar{F}^{-1}(y)p(y)dF(y)\}dF(z)$$

$$- \bar{F}(T)\exp\{\int_0^T \bar{F}^{-1}(y)q(y)dF(y)\}\int_0^T \bar{F}^{-1}(z)q(z)h(z)dF(z).$$

Thus

$$E[R_1^*] = c_\infty F_p(T) + c_0\bar{F}_p(T) + \int_0^T \bar{F}^{-1}(z)q(z)h(z)\ \exp\{-\int_0^z \bar{F}^{-1}(y)p(y)\,dF(y)\}\,dF(z)$$

$$= c_\infty F_p(T) + c_0\bar{F}_p(T) + \int_0^T q(z)h(z)\ \exp\{\int_0^z \bar{F}^{-1}(y)q(y)\,dF(y)\}\,dF(z).$$

REFERENCES

[1] Asher, H. and Feingold, H. (1979). Comments on "Models for reliability of repaired equipment". IEEE Trans. on Reliability R-19.2, 119.

[2] Barlow, R.E. and Hunter, L.C. (1960). Optimum preventive maintenance policies. Operations Res. 8, 90-100.

[3] Barlow, R.E. and Proschan, F. (1965). Mathematical Theory of Reliability. John Wiley and Sons, New York.

[4] Beichelt, F. (1976). A general preventive maintenance policy. Math. Operationsforsch. u. Statist. 7, 927-932.

[5] Block, H.W., Borges, W.S. and Savits, T.H. (1982). Age-dependent minimal repair. University of Pittsburgh Technical Report No. 82-05.

[6] Boland, P.J. (1982). Periodic replacement when minimal repair costs vary with time. Unpublished report.

[7] Boland, P.J. and Proschan, F. (1983). Optimum replacement of a system subject to shocks. Operations Res. 31, 697-704.

[8] Boland, P.J. and Proschan, F. (1982). Periodic replacement with increasing minimal repair costs at failure. Operations Res. 30, 1183-1189.

[9] Ross, S.M. (1970). Applied Probability Models with Optimazation Applications. Holden-Day, San Francisco.

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A general model is introduced which incorporates minimal repair, planned and unplanned replacements, and costs which depend on time. Finite and infinite horizon results are obtained. Various special cases are considered. Furthermore a shock model with general cost structure is considered.

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